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LIMITS.

BY PROF. DE VOLSON WOOD, HOBOKEN, N. J.

THE arguments raised against the infinitesimal Calculus, in the last July ANALYST, have been considered by different writers at various times since the days of Leibnetz; and the philosophy of this method, as well as the Newtonian method of Limits, and of Lagrange's method is clearly set forth in Compte's *Philosophy of Mathematics* (Chap. III, Gillespie's translatson).

The Infinitesimal method is more arbitrary, and hence less inductive and apparently less philosophical, than the method of Limits; still it is none the less a valid *system*. It is not, however, just to that system to use it for a part of our argument and some other system for the remaining part. The argument must be self-consistent and in accordance with the principles of the analysis. We notice one point:—

This method asserts that the finite value of $1 + \circ$ is 1, where the horizontal zero is, according to Prof. Judson's notation, infinitesimal. A *supposed* fallacy in this statement is detected by showing that $[1 + (1 \div x)]^x$ is not 1 for x infinite, whereas all finite powers of 1 are 1; but the infinitesimal system asserts that $\infty \times 0 =$ *something*, and, in this example there will be an infinity of foctors when x is infinite, each of the form $1 + \circ$, and hence it may be possible that $(1 + \circ)^\infty$ will exceed unity; and hence the value found is not inconsistent with the principles of the system.

There is an element in the definition of the limit, which is worthy of notice. The definition given is—"The *limit* of a variable is a *constant* which the variable indefinitely approaches."

"Cor. I. The variable can never reach its limit, otherwise the approach would not be indefinite."

Many writers incorporate the corollary into the definition; and taken together they constitute the substance, and in many cases the language, of the definition of the limit as given by many modern writers. For our part we do not consider the corollary as a necessary consequence of the definition; or if incorporated with the definition it unnecessarily restricts the law of approach of the variable. Todhunter, in his Differential Calculus, asserts that a variable can not reach its limit, and yet, on the same page, asserts that any one of the values of the variable may be considered as a limit. Is it true that a variable cannot reach any one of its values? We admit that a variable may be subjected to such a law that to the human mind it will appear impossible for it to reach the limit, but we assert that it may also be subjected to such a law that it will reach it. Writers generally give illustrations of a law of approach (as, for instance, a descending geometrical progression approaching zero as a limit) according to which we are unable to see how the limit can be reached so long as the law is continuous; but they make no use of the illustration, or principle, in practice.

Indefinite approach may imply that the difference between the variable and its limit shall be less than any assignable quantity; and in this sense the difference may be absolutely zero without doing violence to the definition—in which case the limit will be reached.

But the argument sometimes employed to sustain the definition is open to objection. Thus, if a point moves from A towards B, moving over the first half of the distance in one-half of a minute, one-half of the remaining distance (or $\frac{1}{4}AB$) in one-fourth of a minute, and so on; when will it reach B? Some assert that, according to this law, B can never be reached. Professor Newcomb reasons thus (Alg., p. 212): "since the remainder is halved, if there was any movement that would overcome the entire remainder, the half of a thing would equal the whole, which is impossible". As forcible and apparently unanswerable as this argum't seems to be, it is shown to be fallacious by its proving too much. Thus, by precisely the same argument, it may be proved that the min. hand of a watch can never overtake the hour hand; that two intersecting right lines can never intersect; that bodies at rest can never move, &c.

In the above illustration, we are asked to admit the fundamental difficulty. It is stated that the point moves over one-half the distance, &c.; now explain *how* it moves over half the distance, and we will explain how it reaches B according to the same law.

We assert that there is no more difficulty, and precisely the same difficulty in reaching B that there is in leaving A. It cannot get any distance from A without first having passed over one-half of that distance; and since no distance is so small that it cannot be halved, the *smallest* movement would pass over the smallest space and its half at the sme time which is absurd; therefore it cannot leave A! But motion does occur; hence the logic must be fallacious. The trouble with the logic is, it does not include the *immeasurably* small quantities which make time and space absolutely continuous. Introduce the term *finite*, or *measurable*, and the logic is self consistent. We know nothing of the ultimate elements of time and space; but Newton considered time as *flowing* at a uniform rate, and that finite time may be considered as the sum of all the elements. This is a natural way of considering the subject, and we assert that it is possible to connect a variable to time in such a way that the variable shall reach the limit *under the operation of said law* in a finite time. Thus, for example, if the apothem of an inscribed square should grow in length at a uniform rate, and if polygons of double the number of sides be conceived to be instantly described as the growing line becomes the apothem of an octagon, then of a polygon of sixteen sides, and so on, the inscribed polygons will reach the circle in a finite time.

This is an extreme case. It does not include the possibility of actually constructing the polygons, nor even the possibility of carrying out the conception; but simply that, under the operation of the law, the limit will be reached. But the Calculus does not determine the area of the circle in this way. The abscissa x may be conceived to grow uniformly, and thus to reach any assigned limit between 0 and r; and at the same time the ordinate and area will reach their limit. Definitions should not exclude a kn'n fact. We therefore suggest that the Corollary be excluded, and that the following be substituted for the definition:

The *limit* of a variable is a quantity which the variable approaches and from which it may be made to differ by less than any assignable quantity.

We have here used the term *quantity* instead of *constant* so as to make the definition more general.

In regard to *the law of continuity*, Prof. Price seems to consider it reasonable that it should be endless in extent. Thus an ellipse is a continuous curve, and the function $y = \frac{b}{a}\sqrt{(a^2-x^2)}$, is a continuous function of x, and real from $x = -a$ to $x = +a$. But x, as an independent variable, is continuous and real for all finite values, but the function becomes imaginary for all values of x greater than a, in which case it may be written

$$y = \frac{b}{a}\sqrt{(x^2-a^2)}\sqrt{-1}.$$

The *real* part of the preceding expression is the equation of an hyperbola, and hence the entire expression is called an imaginary hyperbola.

Prof. Price sought such an explanation of *imaginaries* and *infinities* as would make the locus continuous. For this purpose he considered that the effect of $\sqrt{-1}$ was to turn the locus through a right angle perpendicular to the plane of the ellipse and that in that plane the hyperbola would be real. The ellipse and hyperbola will then have a common axis and common vertices, but will be in mutually perpendicular planes. This assumption is arbitrary, but it serves the purpose of preserving *the law of continuity*.

We also find that tan x passes from $+\infty$ to $-\infty$ as x passes $\frac{1}{2}\pi$, hence $-\infty$ is considered as consecutive to $+\infty$. Similarly for other expres'ns.

Admitting that $a \div 0$ is impossible, but that $a \div \circ = \infty$ is rational, we suggest that the former may be expressed thus: $a \div 0 = \wr$. But nothing substantial will be gained by multiplying symbols for this purpose.